A S/V POTENTIAL METHOD FOR MAXWELL'S EQUATIONS NEAR CORNERS USING NODAL ELEMENTS[1]

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Abstract

A new method is presented for obtaining accurate solutions to time harmonic Maxwell's equations when sharp corners are present. It uses a new formulation of the impedance boundary condition which includes the scalar potential in such a way that boundary conditions are satisfied exactly.

1 Summary

Scalar and vector potential methods [1, 2, 3, 4, 5] have been recognized for some time as accurate and rigorous ways of obtaining divergence-free solutions to Maxwell's time harmonic equations utilizing nodal based finite elements. These methods have important advantages over edge element [6] based methods in that they typically require lower sampling rates, resulting in smaller matrices, and can be more easily solved with preconditioned iterative methods, a necessity for large problems.

However, they have suffered from an almost total loss of accuracy whenever sharp metallic corners are present in the model. While applications such as microwave hyperthermia treatment planning and geophysical modeling, which have fruitfully used these methods for some time, do not often require such capabilities, this severely limits their utility in modeling microwave devices and in RCS applications.

Attempts to eliminate this errant behavior have primarily involved hybrid nodal/edge ele-

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ments [7, 8] and have met with limited success. The use of these hybrid elements was motivated by the observation that the vector potential alone cannot satisfy the proper boundary conditions when expressed solely with nodal element basis functions. The addition of correctly chosen edge element basis functions, though not rigorous, did improve the results.

We present a new treatment for this phenomena which requires only second order nodal basis functions. It incorporates both the scalar and vector potentials into an impedance boundary condition (SV-IBC) that permits the boundary conditions on the electric field to be satisfied exactly.

2 SV-IBC

When scalar and vector potentials

$$\vec{E} = \imath\omega\vec{A} - \nabla\phi, \tag{1}$$

are introduced into the time harmonic Maxwell's equations a redundancy is created. For if $(\vec{A}, \phi)$ and $(\vec{B}, \psi)$ are two such potential pairs whose electric field solves the same electromagnetic problem, then $\vec{A} = \vec{B} + \frac{1}{\imath\omega}\nabla(\phi - \psi)$. That is, ϕ and ψ could differ by a constant without affecting $\vec{A}$, $\vec{B}$ or $\vec{E}$ and ϕ and ψ could be arbitrary without affecting $\vec{E}$.

Thus, additional boundary conditions are required to obtain a unique solution. These conditions may be viewed as necessary conditions to uniquely determine the scalar potential, forcing uniqueness on the vector potential. Typical additional boundary conditions [1, 2, 3] involve specifying $\phi = 0$ or $\partial\phi/\partial n = 0$ since ϕ satisfies a scalar Helmholtz or Poisson equation.

For perfect electric and magnetic conductors, the proper conditions are

$$\begin{aligned} \hat{n} \times \left(\imath\omega\vec{A} - \nabla\phi\right) &= 0, && (2)\\ \nabla \cdot \vec{A} - \imath\omega\epsilon\mu\phi &= 0, && (3)\\ \phi &= 0, && (4) \end{aligned}$$

and

$$\begin{aligned} \hat{n} \times \frac{1}{\mu}\nabla \times \vec{A} &= 0, && (5)\\ \hat{n} \cdot \epsilon\left(\imath\omega\vec{A} - \nabla\phi\right) &= 0, && (6)\\ \frac{\partial\phi}{\partial n} &= 0, && (7) \end{aligned}$$

respectively. Note that both conditions require the explicit use of the normal to the surface, a quantity undefined at a sharp corner. This is the root of the problems that nodal methods have with sharp PEC corners.

Consider the PEC condition (2,3,4). Since $\phi = 0$, $\hat{n} \times \nabla\phi = 0$ and thus the vector potential $\vec{A}$ must be normal to the surface. This is an essential boundary condition and is normally enforced at each node along the surface. If the surface is smooth and not highly curved (relative to element size) $\vec{A}$ will also be approximately normal everywhere on the PEC. Using this condition for objects with corners, with an "average" normal at corners, leads to complete loss of accuracy [8].

Using an impedance boundary condition with small impedance and $\phi = 0$, which is enforced only weakly on the surface, avoids explicit use of the non-existant normal at the corner, but does no better in regards to accuracy [8]. This is due to the fact that the nodal basis representation for $\vec{A}$ (and thus $\vec{E}$) in this case does not permit the PEC boundary conditions to be even approximately satisfied on both sides of a corner.

However, if the scalar potential is not set to zero on the PEC surface, it provides additional degrees of freedom with which to force tangential $\vec{E}$ to vanish 2. This is the idea behind the SV-IBC. If in addition, the vector potential is unconstrained at corner nodes and the mid-side nodes on the PEC adjacent to those corner nodes, and ϕ is set to zero at an arbitrary point on the PEC, then the SV-IBC with small impedance forces the PEC conditions to be satisfied exactly on the surface and results in a stable non-singular system. This has been verified mathematically and numerically with eigenvalue analysis of the resulting finite element matrices.

Numerical examples were run for a 2-D PEC body with a sharp corner. Comparisons were made between a scalar and vector potential formulation for (E_x, E_y) and the curl of a scalar formulation for H_z. The results at centroids of the finite elements are indistinguishable from one another, showing only a few percent RMS error. The behavior of the scalar and vector potential on the PEC near a corner is the most impressive.

Figure 1 shows the vector potential near a corner of the body. Note that it does not (and cannot) be normal on both sides of the corner. Figure 2 shows the corresponding plot of $\nabla\phi$, while Fig. 3 shows the resulting electric field. Note how the electric field is perfectly normal on both sides of the corner.

Similar results have been obtained with both

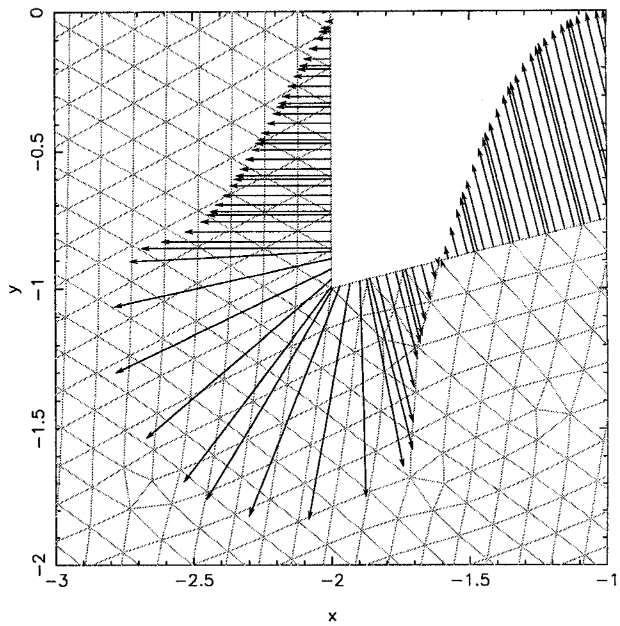

Figure 1: Vector Potential: $\imath\omega\vec{A}$

electric and magnetic field formulations and small/large/intermediate values of impedance, all producing accurate answers indistinguishable from the benchmark results.

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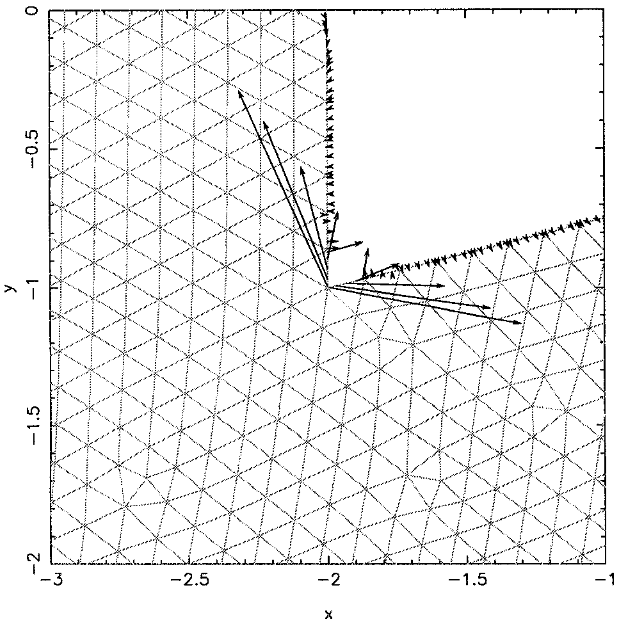

Figure 2: Scalar Potential: $-\nabla\phi$

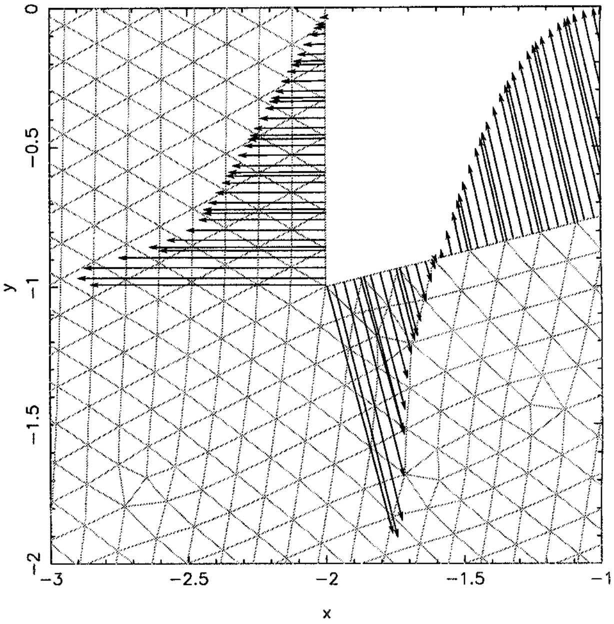

Figure 3: $\vec{E} = \imath\omega\vec{A} - \nabla\phi$

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